A QUANTUM STOCHASTIC LIE-TROTTER PRODUCT FORMULA

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Abstract. A Trotter product formula is established for unitary quantum stochastic processes governed by quantum stochastic differential equations with constant bounded coefficients.

Introduction

The aim of this paper is to establish a quantum probabilistic counterpart to the well-known Trotter product formula for one-parameter unitary groups and contraction semigroups ([Tro]) and its forerunner, the Lie product formula for one-parameter subgroups of Lie groups (see [Dav], [$\mathrm{RS}_{1,2}$]. Some years ago K.R. Parthasarathy and the second-named author obtained a stochastic Trotter product formula for unitary-operator valued evolutions constituted from independent increments of independent classical Brownian motions ([PS_1]). This predated the founding of quantum stochastic calculus by Hudson and Parthasarathy ([HuP]). In this paper Brownian increments are replaced by the fundamental quantum martingales, namely the creation, preservation and annihilation processes of quantum stochastic calculus ([Bia], [Hud], [L], [Mey], [Par], [SiG]), and we prove a Lie-Trotter type product formula for unitary quantum stochastic processes on a Hilbert space which satisfy a quantum stochastic differential equation with constant bounded coefficients. The case of quantum stochastic differential equations with unbounded coefficients, and more general kinds of quantum stochastic cocycle on operator spaces and C^*-algebras, will be addressed in the forthcoming paper [LS_2].

1. Unitary quantum stochastic cocycles

In this section we fix our notations and recall the essential facts about quantum stochastic differential equations and unitary quantum stochastic cocycles that we need here.

Let k be a complex Hilbert space, with fixed countable orthonormal basis, which we refer to as the *noise dimension space*. Write $\mathcal{F}_{\mathsf{k}}$ for the symmetric Fock space over the Hilbert space $\mathsf{K} := L^2(\mathbb{R}_+; \mathsf{k})$ and

2000 *Mathematics Subject Classification.* Primary 81S25; Secondary 46L53, 47D06.

Key words and phrases. Trotter product formula, quantum stochastic cocycle, one-parameter semigroup, random unitary evolution, noncommutative probability.

$\varpi(f)$ for the normalised exponential vector $\exp(-\|f\|^2/2)\,\varepsilon(f)$, $f \in \mathsf{K}$. When $\mathbb{R}_+$ is replaced by $[s,t[$, we write $\mathsf{K}_{[s,t[}$ and $\mathcal{F}_{\mathsf{k},[s,t[}$ instead; the continuous tensor decomposition

$$\mathcal{F}_{\mathsf{k}} = \mathcal{F}_{\mathsf{k},[0,s[} \otimes \mathcal{F}_{\mathsf{k},[s,t[} \otimes \mathcal{F}_{\mathsf{k},[t,\infty[},$$

corresponding to the direct sum decomposition $\mathsf{K} = \mathsf{K}_{[0,s[} \oplus \mathsf{K}_{[s,t[} \oplus \mathsf{K}_{[t,\infty[}$, is in constant use below. For a start, a bounded quantum stochastic process on an *initial* Hilbert space $\mathfrak{h}$ with noise dimension space k is a family of operators $(X_t)_{t\geq 0}$ on $\mathfrak{h} \otimes \mathcal{F}_{\mathsf{k}}$ satisfying the adaptedness condition

$$X_t \in B(\mathfrak{h} \otimes \mathcal{F}_{\mathsf{k},[0,t[}) \otimes I_{\mathcal{F}_{\mathsf{k},[t,\infty[}} = B(\mathfrak{h}) \overline{\otimes} B(\mathcal{F}_{\mathsf{k},[0,t[}) \otimes I_{\mathcal{F}_{\mathsf{k},[t,\infty[}}, \quad (1.1)$$

for all $t \in \mathbb{R}_+$. For $f \in \mathsf{K}$, $f_{[s,t[}$ denotes the function equal to f on $[s,t[$ and zero elsewhere; $c_{[s,t[}$ is defined similarly, for $c \in \mathsf{k}$. Let $\mathbb{S}_{\mathsf{k}}$ and $\mathbb{S}'_{\mathsf{k}}$ denote the subspaces of K consisting of step functions, respectively step functions which have their discontinuities in the dyadic set $\mathbb{D} := \{j2^{-n} : j,n \in \mathbb{Z}_+\}$, and let $\mathcal{E}_{\mathsf{k}}$ and $\mathcal{E}'_{\mathsf{k}}$ be the (dense) subspaces $\mathrm{Lin}\{\varepsilon(f) : f \in \mathbb{S}_{\mathsf{k}}\}$ and $\mathrm{Lin}\{\varepsilon(f) : f \in \mathbb{S}'_{\mathsf{k}}\}$ of $\mathcal{F}_{\mathsf{k}}$. For evaluation purposes, we always take the *right-continuous versions* of step functions. The *order* of a function $f \in \mathbb{S}'_{\mathsf{k}}$ is the least nonnegative integer N such that f is constant on all intervals of the form $[j2^{-N},(j+1)2^{-N}[$ for $j \in \mathbb{Z}_+$.

The *time-shift* semigroup $(\Theta^{\mathsf{k}}_t)_{t\geq 0}$ of unital *-monomorphisms of $B(\mathcal{F}_{\mathsf{k}})$ is defined by

$$\Theta^{\mathsf{k}}_t(X) = I_{\mathcal{F}_{\mathsf{k},[0,t[}} \otimes \Gamma(\theta^{\mathsf{k}}_t) X \Gamma(\theta^{\mathsf{k}}_t)^*, \quad t \in \mathbb{R}_+, X \in B(\mathcal{F}_{\mathsf{k}}),$$

where $\Gamma(\theta^{\mathsf{k}}_t) : \mathcal{F}_{\mathsf{k}} \to \mathcal{F}_{\mathsf{k},[t,\infty[}$ is the unitary (second quantisation) operator determined by

$$\Gamma(\theta^{\mathsf{k}}_t)\varpi(f) = \varpi(\theta^{\mathsf{k}}_t f) \text{ where } (\theta^{\mathsf{k}}_t f)(s) = f(s-t) \text{ for } s \in [t,\infty[.$$

Let $\{\Lambda^\mu_\nu : \mu,\nu \geq 0\}$ denote the fundamental quantum semimartingales for the noise dimension space k, with respect to its fixed orthonormal basis. Then the quantum stochastic (QS) integral equation

$$U_t = I_{\mathfrak{h}\otimes\mathcal{F}_{\mathsf{k}}} + \int_0^t U_s F^\mu_\nu \Lambda^\nu_\mu(ds) \quad (1.2)$$

(where summation over the repeated greek indices is understood), has a unique strongly continuous solution, consisting of unitary operators on $\mathfrak{h} \otimes \mathcal{F}_{\mathsf{k}}$, provided that the matrix of bounded operators $[F^\mu_\nu]$ on the initial space $\mathfrak{h}$ satisfies the following structural relations ([HuP]). It must have the block matrix structure

$$\begin{bmatrix} K & [M_k] \\ [L^j] & [W^j_k - \delta^j_k] \end{bmatrix}$$

of an operator $F \in B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}))$, where $[L^j]$ is the block column matrix of an *arbitrary* operator $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$, $[W^j_k]$ is the block matrix form of a *unitary* operator $W \in B(\mathfrak{h} \otimes \mathsf{k})$, $[M_k]$ is the block row matrix of *the* operator $M = -L^*W \in B(\mathfrak{h} \otimes \mathsf{k}; \mathfrak{h})$, and $K = iH - \frac{1}{2}L^*L$ for a *selfadjoint* operator $H \in B(\mathfrak{h})$, so that

$$M_k = -\sum_{j\geq 1}(L^j)^*W^j_k,\ k \geq 0,\ \text{and}\ K = iH - \frac{1}{2}\sum_{j\geq 1}(L^j)^*L^j.$$

These structure relations may equivalently be expressed by the following two identities, for all $v = (v^\mu)_{\mu\geq 0}$ in $\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}) = \bigoplus_{\mu\geq 0}\mathfrak{h}$:

$$\sum_{\mu,\nu\geq 0}\big\langle v^\mu, \big((F^\nu_\mu)^* + F^\mu_\nu + \sum\nolimits_{j\geq 1}(F^j_\mu)^*F^j_\nu\big)v^\nu\big\rangle = 0, \tag{1.3a}$$

$$\sum_{\mu,\nu\geq 0}\big\langle v^\mu, \big((F^\nu_\mu)^* + F^\mu_\nu + \sum\nolimits_{j\geq 1}F^\mu_j(F^\nu_j)^*\big)v^\nu\big\rangle = 0; \tag{1.3b}$$

the first corresponds to isometry and the second to coisometry.

A contractive quantum stochastic process $(U_t)_{t\geq 0}$ satisfying

$$U_{s+t} = U_s\Theta_s(U_t),\quad U_0 = I_{\mathfrak{h}\otimes\mathcal{F}},\quad s,t \geq 0, \tag{1.4}$$

where $\big(\Theta_t := \mathrm{id}_{B(\mathfrak{h})}\,\overline{\otimes}\,\Theta^{\mathsf{k}}_t\big)_{t\geq 0}$, is called a *quantum stochastic contraction cocycle*. If $(U_t)_{t\geq 0}$ is a QS contraction cocycle then the operators on $\mathfrak{h}$ defined by

$$\langle u, P_t v\rangle = \langle u \otimes \varpi(0), U_t\, v \otimes \varpi(0)\rangle \quad u, v \in \mathfrak{h},\ t \in \mathbb{R}_+,$$

define a contraction semigroup $(P_t)_{t\geq 0}$ on $\mathfrak{h}$ known as the (*vacuum*) *expectation semigroup* of the cocycle, and the cocycle $(U_t)_{t\geq 0}$ is called *Markov-regular* if its expectation semigroup is norm-continuous.

Theorem 1.1 ([LW$_1$])**.** *Let $(U_t)_{t\geq 0}$ be a unitary quantum stochastic process on $\mathfrak{h}$ with noise dimension space k. Then the following are equivalent:*

(i) $(U_t)_{t\geq 0}$ *satisfies* (1.2)*, for a matrix of bounded operators* $[F^\mu_\nu]$*;*
(ii) $(U_t)_{t\geq 0}$ *is a Markov-regular quantum stochastic cocycle.*

The implication (i) ⇒ (ii) follows from the form that solutions of such QS differential equations take, by virtue of the time-homogeneity of the quantum noises:

$$I_{\mathcal{F}_{\mathsf{k},[0,t[}} \otimes \Gamma(\theta^{\mathsf{k}}_t)\Lambda^\mu_\nu[a,b]\Gamma(\theta^{\mathsf{k}}_t)^* = \Lambda^\mu_\nu[a+t, b+t],$$

and the time-independence of the coefficients of the QS differential equation.

The converse implication (ii) ⇒ (i) may be deduced from the Quantum Martingale Representation Theorem ([PS$_2$]) applied to the regular quantum martingale

$$\Big(U_t - \int_0^t U_sK\,ds\Big)_{t\geq 0}$$

in which the operator K is the generator of the expectation semigroup of $(U_t)_{t\geq 0}$ (see [HuL]). However the more powerful method of proof in [LW$_1$] goes via the following intermediate characterisation which is of considerable use itself, as we shall see below:

(iii) there are semigroups $\{(P^{c,d})_{t\geq 0} : c,d \in \mathsf{k}\}$ such that, for all $f,g \in \mathbb{S}'_{\mathsf{k}}$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}\langle u \otimes \varpi(f_{[0,t[}), V_t\, v \otimes \varpi(g_{[0,t[})\rangle &= \\ &\langle u, P^{f(t_0),g(t_0)}_{t_1-t_0} \cdots\ P^{f(t_m),g(t_m)}_{t_{m+1}-t_m} v\rangle,\end{aligned} \tag{1.5}$$

where $t_0 = 0$, $t_{m+1} = t$ and $\{t_1 < \cdots < t_m\} \subset \mathbb{D}$ is the (possibly empty) union of the sets of discontinuity of f and g in the open interval $]0,t[$.

Remarks. The matrix of bounded operators $[F^\mu_\nu]$ necessarily satisfies the structural relations required for unitarity (1.3).

The identity (1.5) is known as the *semigroup decomposition* and the collection $\{(P^{c,d}_t)_{t\geq 0} : c,d \in \mathsf{k}\}$ as the *associated semigroups* of the cocycle. Clearly the associated semigroups are determined by

$$\langle u, P^{c,d}_t v\rangle = \langle u \otimes \varpi(c_{[0,t[}), U_t\, v \otimes \varpi(d_{[0,t[})\rangle, \quad u,v \in \mathfrak{h}, \tag{1.6}$$

and $(P^{0,0})_{t\geq 0}$ is the expectation semigroup of the cocycle.

In fact, each associated semigroup $(P^{c,d})_{t\geq 0}$ is itself the expectation semigroup of another unitary QS cocycle, namely the cocycle

$$\Big(U^{c,d}_t := (I_{\mathfrak{h}} \otimes W^c_t)^*\, U_t (I_{\mathfrak{h}} \otimes W^d_t)\Big)_{t\geq 0},$$

where the *Weyl cocycles* are defined by

$$W^c_t \varpi(f) = e^{-i\,\mathrm{Im}\langle c_{[0,t[}, f\rangle} \varpi(f + c_{[0,t[}), \quad f \in \mathbb{S}_{\mathsf{k}},\ c \in \mathsf{k},\ t \in \mathbb{R}_+.$$

Markov-regularity for a QS contraction cocycle actually implies that all of its associated semigroups are norm-continuous. In fact, in terms of the block matrix form of $[F^\mu_\nu]$, the semigroup $(P^{c,d})_{t\geq 0}$ has bounded generator

$$G_{c,d} := K + L^c + M_d + W^c_d - \tfrac{1}{2}(\|c\|^2 + \|d\|^2) I_{\mathfrak{h}}, \tag{1.7}$$

where, in terms of basis expansions of c and d, the operators here are defined as follows:

$$L^c = \sum_{j\geq 1} \overline{c^j} L^j, \quad M_d = \sum_{k\geq 1} d^k M_k \text{ and } W^c_d = \sum_{j,k\geq 1} \overline{c^j} d^k W^j_k,$$

the convergence here being in the strong operator topology (see [LW$_2$]).

Given a unitary QS cocycle $(U_t)_{t\geq 0}$, the family $(U_{s,t} := \Theta_s(U_{(t-s)}))_{0\leq s\leq t}$ is a *time-homogeneous adapted unitary evolution*, that is: for all $a \geq 0$ and $0 \leq r \leq s \leq t$:

(i) $U_{s+a,t+a} = \Theta_a(U_{s,t})$;
(ii) $U_{s,t} \in B(\mathfrak{h}) \otimes I_{\mathcal{F}_{\mathsf{k},[0,t[}} \overline{\otimes}\, B(\mathcal{F}_{\mathsf{k},[s,t[}) \otimes I_{\mathcal{F}_{\mathsf{k},[t,\infty[}}$;
(iii) $U_{r,t} = U_{r,s}\, U_{s,t}$.

Conversely, if $(U_{s,t})_{0\le s\le t}$ is such an evolution then $(U_t := U_{0,t})_{t\ge 0}$ defines a unitary QS cocycle, and it is easily seen that the passages between QS cocycle and adapted time-homogeneous evolution are mutually inverse.

The corresponding QS integral equation satisfied by $(U_{s,t})_{0\le s\le t}$ is

$$U_{r,t} = I_{\mathfrak{h}\otimes\mathcal{F}_{\mathsf{k}}} + \int_r^t U_{r,s} F^\mu_\nu \Lambda^\nu_\mu(ds).$$

Adapted evolutions that are not time-homogeneous arise as solutions of QS differential equations with time-dependent coefficients $[F^\mu_\nu]$.

2. Trotter product of quantum stochastic cocycles

Let $(U^1_t)_{t\ge 0}$ and $(U^2_t)_{t\ge 0}$ be two unitary QS cocycles on the same initial space $\mathfrak{h}$, with noise dimension spaces k_1 and k_2 having fixed countable orthonormal bases. Suppose that they are both Markov-regular, equivalently that they satisfy QS differential equations

$$dU^l_t = U^l_s \, {}^{(l)}F^{\mu_l}_{\nu_l} \, \Lambda^{\nu_l}_{\mu_l}(dt), \quad U^l_0 = I_{\mathfrak{h}\otimes\mathcal{F}^{(l)}}, \tag{2.1}$$

$l = 1, 2$, for matrices of bounded operators $[{}^{(1)}F^{\mu_1}_{\nu_1}]$ and $[{}^{(2)}F^{\mu_2}_{\nu_2}]$ satisfying the structural relations which guarantee unitarity of the processes. Here $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ denote the Fock spaces $\mathcal{F}_{\mathsf{k}_1}$ and $\mathcal{F}_{\mathsf{k}_2}$ respectively.

Our aim is to obtain a unitary cocycle $(U_t)_{t\ge 0}$ as a Lie-Trotter type product of the cocycles $(U^1_t)_{t\ge 0}$ and $(U^2_t)_{t\ge 0}$, in the same spirit as that of [PS$_1$]. To this end let k be the noise dimension space $\mathsf{k}_1 \oplus \mathsf{k}_2$, set $\mathcal{F} = \mathcal{F}_{\mathsf{k}}$, and, by 'concatenating' the orthonormal bases for k_1 and k_2 to form an orthonormal basis of k, let $[F^\mu_\nu]$ be the matrix of bounded operators on $\mathfrak{h}$ having block matrix form

$$\begin{bmatrix} K & [M_k] \\ [L^j] & [W^j_k - \delta^j_k I_{\mathfrak{h}}] \end{bmatrix} = \begin{bmatrix} {}^{(1)}K + {}^{(2)}K & {}^{(1)}M & {}^{(2)}M \\ {}^{(1)}L & {}^{(1)}W - {}^{(1)}I & 0 \\ {}^{(2)}L & 0 & {}^{(2)}W - {}^{(2)}I \end{bmatrix}. \tag{2.2}$$

Here ${}^{(l)}I := I_{\mathfrak{h}\otimes\mathsf{k}_l}$ and

$$[{}^{(l)}F^{\mu_l}_{\nu_l}] = \begin{bmatrix} {}^{(l)}K & [{}^{(l)}M_{k_l}] \\ [{}^{(l)}L^{j_l}] & [{}^{(l)}W^{j_l}_{k_l} - \delta^{j_l}_{k_l} I_{\mathfrak{h}}] \end{bmatrix}$$

is the block matrix decomposition of ${}^{(l)}F$, in which

$${}^{(l)}K = iH_l - \frac{1}{2}\sum_{j_l\ge 1} ({}^{(l)}L^{j_l})^* \, {}^{(l)}L^{j_l} \quad \text{and } (H_l)^* = H_l,$$

for $l = 1, 2$. (We are slightly cheating in terms of indices since if the noise dimension space k_1 is infinite dimensional then we cannot *exactly* count $1, 2, \cdots, \dim \mathsf{k}_1, \dim \mathsf{k}_1 + 1, \cdots$. However all *is* justified by a proper indexing, or alternatively by working coordinate-free as in [LS$_2$].) Thus, setting $H = H_1 + H_2$,

$$\begin{aligned} K &= iH - \frac{1}{2}\sum_{j\geq 1}(L^j)^* L^j \\ &= iH_1 + iH_2 - \frac{1}{2}\sum_{j_1\geq 1}({}^{(1)}L^{j_1})^*({}^{(1)}L^{j_1}) - \frac{1}{2}\sum_{j_2\geq 1}({}^{(2)}L^{j_2})^*({}^{(2)}L^{j_2}), \text{ and} \end{aligned}$$

$$[M_k] = \begin{bmatrix} -\,{}^{(1)}L^*\,{}^{(1)}W & -\,{}^{(2)}L^*\,{}^{(2)}W \end{bmatrix} = \begin{bmatrix} -\sum_{j\geq 1}(L^j)^* W^j_k \end{bmatrix}.$$

Thus $[F^\mu_\nu]$ satisfies the structure relations (1.3) for unitarity of the solution of the QS differential equation (1.2).

For $c^l, d^l \in \mathsf{k}_l$, let $({}^{(l)}P^{c^l,d^l}_t)_{t\geq 0}$ denote the corresponding associated semigroup of the cocycle $(U^l_t)_{t\geq 0}$ $(l = 1, 2)$. For each $n \in \mathbb{N}$ define a unitary process $(U^{(1,2)}_n(t))_{t\geq 0}$ as follows:

$$U^{(1,2)}_n(t) := \big(U^{(1,2)}_{0,2^{-n}} U^{(1,2)}_{2^{-n},2\cdot 2^{-n}} \cdots U^{(1,2)}_{t^n_{-1},t^n_0}\big) U^{(1,2)}_{t^n_0,t}, \quad t \in \mathbb{R}_+,$$

where, with $[\,\cdot\,]$ denoting the integer part,

$$t^n_k := 2^{-n}\big([2^n t] + k\big) \qquad \text{for } k \in \mathbb{Z}, k \geq -[2^n t], \tag{2.3}$$

and, letting $\Sigma_{2,1}$ denote the tensor flip $B(\mathfrak{h} \otimes \mathcal{F}^{(2)} \otimes \mathcal{F}^{(1)}) \to B(\mathfrak{h} \otimes \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}) = B(\mathfrak{h} \otimes \mathcal{F})$,

$$U^{(1,2)}_{s,t} := \Theta_s(U^{(1,2)}_{t-s}), \qquad 0 \leq s \leq t, \tag{2.4}$$

where

$$U^{(1,2)}_t := \big(U^1_t \otimes I^{(2)}\big)\Sigma_{2,1}\big(U^2_t \otimes I^{(1)}\big), \quad t \in \mathbb{R}_+.$$

Here $I^{(l)}$ is the identity operator on $\mathcal{F}^{(l)}$ $(l = 1, 2)$, and we are using the isometric isomorphism $\mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} = \mathcal{F}$. Also define a family of contractions on $\mathfrak{h}$ by

$$\langle u, {}^{(1,2)}P^{c,d}_t v\rangle = \langle u \otimes \varpi(c_{[0,t[}), U^{(1,2)}_t\, v \otimes \varpi(d_{[0,t[})\rangle, \quad u, v \in \mathfrak{h},$$

for $c, d \in \mathsf{k}$ and $t \in \mathbb{R}_+$ (cf. (1.6)).

Remarks. For each $t \in \mathbb{R}_+$, $n \in \mathbb{N}$ and k as in (2.3),

$$t^n_0 \leq t^{n+1}_0 \leq t < t^{n+1}_1 < t^n_1 \text{ and } |t^n_{k+1} - t^n_k| = 2^{-n}.$$

In particular the sequence (t^n_1) decreases to t and the sequence (t^n_0) is nondecreasing and converges to t.

In general, neither $(U^{(1,2)}_t)_{t\geq 0}$ nor $(U^{(1,2)}_n(t))_{t\geq 0}$ are cocycles themselves. However they are both unitary QS processes and the two-parameter process $(U^{(1,2)}_{s,t})_{0\leq s\leq t}$ enjoys the factorisations

$$U^{(1,2)}_{s,t} \in B(\mathfrak{h}) \otimes I_{[0,s[}\overline{\otimes}\, B(\mathcal{F}_{[s,t[}) \otimes I_{[t,\infty[} \tag{2.5}$$

in which $I_{[0,s[}$ and $I_{[t,\infty[}$ denote the identity operators on $\mathcal{F}_{[0,s[}$ and $\mathcal{F}_{[t,\infty[}$. By the same token, $({}^{(1,2)}P^{c,d})_{t\geq 0}$ is typically not a semigroup.

Lemma 2.1. *Let $(U^1_t)_{t\ge 0}$ and $(U^2_t)_{t\ge 0}$ be unitary QS cocycles on $\mathfrak{h}$ with noise dimension spaces k_1 and k_2 respectively. Set $\mathsf{k} := \mathsf{k}_1 \oplus \mathsf{k}_2$ and let $(U^{(1,2)}_t)_{t\ge 0}$ be as defined above. Let $t \in \mathbb{R}_+$, then for $c = \binom{c^1}{c^2}$, $d = \binom{d^1}{d^2} \in \mathsf{k} = \mathsf{k}_1 \oplus \mathsf{k}_2$,*

$$ {}^{(1,2)}P^{(c,d)}_t = {}^{(1)}P^{c^1,d^1}_t \, {}^{(2)}P^{c^2,d^2}_t, $$

and, for $f, g \in \mathbb{S}'_{\mathsf{k}}$ and n greater than the orders of both f and g,

$$ \langle u \otimes \varpi(f_{[0,t[}), U^{(1,2)}_n(t)\, v \otimes \varpi(g_{[0,t[}) \rangle = $$
$$ \Big\langle u, \big({}^{(1,2)}P^{f(0),g(0)}_{2^{-n}} \, {}^{(1,2)}P^{f(2^{-n}),g(2^{-n})}_{2^{-n}} \cdots {}^{(1,2)}P^{f(t^n_{-1}),g(t^n_{-1})}_{2^{-n}} \big) {}^{(1,2)}P^{f(t^n_0),g(t^n_0)}_{(t-t^n_0)} \, v \Big\rangle. $$

Proof. These both follow from factorisations; the first from

$$ \big\langle u \otimes \varpi(c_{[0,t[}), U^{(1,2)}_t \, v \otimes \varpi(d_{[0,t[}) \big\rangle = $$
$$ \Big\langle u \otimes \varpi(c^1_{[0,t[}), U^1_t \big((E^* U^2_t F\, v) \otimes \varpi(d^1_{[0,t[}) \big) \Big\rangle, $$

where E and F are the isometric operators $\mathfrak{h} \to \mathfrak{h} \otimes \mathcal{F}^{(2)}$ defined respectively by $v \mapsto v \otimes \varpi(c^2_{[0,t[})$ and $v \mapsto v \otimes \varpi(d^2_{[0,t[})$; in turn, the second from (2.5) and $\varpi(h) = \varpi(h_{[0,s[}) \otimes \varpi(h_{[s,t[}) \otimes \varpi(h_{[t,\infty[})$, for $h = f, g$. $\square$

We now come to our quantum stochastic product formula. For its proof we use the following version of the classical Lie product formula. For bounded operators Z_1 and Z_2 on $\mathfrak{h}$,

$$ \big(e^{hZ_1} e^{hZ_2}\big)^{[t/h]} \to e^{t(Z_1+Z_2)} \text{ as } h \to 0, \tag{2.6} $$

in operator norm, uniformly on bounded time intervals (see e.g. Theorem VIII.29 of [RS$_1$], where the proof is obviously valid for operators).

Theorem 2.2. *Let $(U^1_t)_{t\ge 0}$ and $(U^2_t)_{t\ge 0}$ be unitary QS cocycles on $\mathfrak{h}$ with noise dimension spaces k_1 and k_2, satisfying the quantum stochastic differential equations* (2.1)*, and let $(U_t)_{t\ge 0}$ be the unitary QS cocycle on $\mathfrak{h}$ with noise dimension space $\mathsf{k} := \mathsf{k}_1 \oplus \mathsf{k}_2$ satisfying the QS differential equation* (1.2) *where $[F^\mu_\nu]$ is given by* (2.2)*. Then,*

$$ U^{(1,2)}_n(t) \to U_t \text{ as } n \to \infty, \tag{2.7} $$

in the strong operator topology on $B(\mathfrak{h} \otimes \mathcal{F})$, for each $t \ge 0$.

Proof. Let $t \in \mathbb{R}_+$. First note that, since U_t is unitary and each $U^{(1,2)}_n(t)$ is unitary and so a contraction, it suffices to prove that $U^{(1,2)}_n(t) \to U_t$ in the weak operator topology. Also, the uniform boundedness of the operators $U^{(1,2)}_n(t)$ means that it suffices to fix $u, v \in \mathfrak{h}$ and $f = \binom{f^1}{f^2}, g = \binom{g^1}{g^2} \in \mathbb{S}'_{\mathsf{k}} \subset L^2(\mathbb{R}_+; \mathsf{k} = \mathsf{k}_1 \oplus \mathsf{k}_2)$, and prove the following:

$$ \big\langle u \otimes \varpi(f_{[0,t[}), U^{(1,2)}_n(t)\, v \otimes \varpi(g_{[0,t[}) \big\rangle \to $$
$$ \big\langle u \otimes \varpi(f_{[0,t[}), U_t \, v \otimes \varpi(g_{[0,t[}) \big\rangle. \tag{2.8} $$

By the semigroup representation (1.5),

$$\text{R.H.S. of (2.8)} = \langle u, P_{t_1-t_0}^{f(t_0),g(t_0)} \cdots P_{t_{m+1}-t_m}^{f(t_m),g(t_m)} v\rangle, \tag{2.9}$$

where $t_0 = 0$, $t_{m+1} = t$ and $\{t_1 < \cdots < t_m\} \subset \mathbb{D}$ are the points in $]0, t[$ (if any) where f or g has a discontinuity. For n greater than the orders of the step functions f and g, Lemma 2.1 implies that the L.H.S. of (2.8) equals

$$\Big\langle u, \big({}^{(1)}P_{2^{-n}}^{f^1(t_0),g^1(t_0)}\, {}^{(2)}P_{2^{-n}}^{f^2(t_0),g^2(t_0)}\big)^{[2^n(t_1-t_0)]} \cdots$$
$$\cdots \big({}^{(1)}P_{2^{-n}}^{f^1(t_m),g^1(t_m)}\, {}^{(2)}P_{2^{-n}}^{f^2(t_m),g^2(t_m)}\big)^{[2^n(t_{m+1}-t_m)]}$$
$$\big({}^{(1)}P_{(t-t_0^n)}^{f^1(t_m),g^1(t_m)}\, {}^{(2)}P_{(t-t_0^n)}^{f^2(t_m),g^2(t_m)}\big)\, v\Big\rangle.$$

The Lie product formula (2.6) and the joint continuity of operator composition on bounded sets, therefore implies that

$$\lim_{n\to\infty} (\,\text{L.H.S. of (2.8)}\,) = \langle u, Q_{t_1-t_0}^{f(t_0),g(t_0)} \cdots Q_{t_{m+1}-t_m}^{f(t_m),g(t_m)} v\rangle, \tag{2.10}$$

where $(Q_t^{c,d})_{t\geq 0}$ is the semigroup generated by $G^{(1)}_{c^1,d^1} + G^{(2)}_{c^2,d^2}$. Now

$$\begin{aligned} G^{(1)}_{c^1,d^1} + G^{(2)}_{c^2,d^2} = \\ &{}^{(1)}K + {}^{(1)}L^{c^1} + {}^{(1)}M_{d^1} + {}^{(1)}W^{c^1}_{d^1} - \tfrac{1}{2}(\|c^1\|^2 + \|d^1\|^2)I_{\mathfrak{h}} \\ &+ {}^{(2)}K + {}^{(2)}L^{c^2} + {}^{(2)}M_{d^2} + {}^{(2)}W^{c^2}_{d^2} - \tfrac{1}{2}(\|c^2\|^2 + \|d^2\|^2)I_{\mathfrak{h}} \\ &= K + L^c + M_d + W^c_d - \tfrac{1}{2}(\|c\|^2 + \|d\|^2)I_{\mathfrak{h}}, \end{aligned}$$

which, by (1.7), is the generator of the semigroup $(P_t^{c,d})_{t\geq 0}$, for each $c, d \in \mathsf{k}$. The result therefore follows from (2.10) and (2.9). □

Remark. The joint continuity of operator composition on bounded sets also gives a straightforward extension of this result to time-homogeneous adapted unitary evolutions $(U_{s,t})_{0\leq s\leq t}$:

$$U_n^{(1,2)}(s,t) \to U_{s,t} \quad \text{as } n \to \infty,$$

in the strong operator topology, for all $0 \leq s \leq t$, where

$$U_n^{(1,2)}(s,t) := U^{(1,2)}_{s,s_1^n}\big(U^{(1,2)}_{s_1^n,s_2^n} U^{(1,2)}_{s_2^n,s_3^n} \cdots U^{(1,2)}_{t_{-1}^n,t_0^n}\big) U^{(1,2)}_{t_0^n,t}.$$

3. Extensions and an example

The quantum stochastic product formula also holds for Markov-regular QS *contraction* cocycles, with the same proof, since these are equally characterised as contraction processes which satisfy a QS differential equation of the form (1.2), in other words Theorem 1.1 still holds; contractivity of the cocycle corresponds precisely to the matrix of coefficients of the QS differential equation satisfying the inequality

$$\sum_{\mu,\nu\geq 0} \big\langle v^\mu, \big((F^\nu_\mu)^* + F^\mu_\nu + \textstyle\sum_{j\geq 1}(F^j_\mu)^* F^j_\nu\,\big) v^\nu\big\rangle \;\leq 0,$$

equivalently,

$$\sum_{\mu,\nu\geq 0} \Big\langle v^\mu, \big((F^\nu_\mu)^* + F^\mu_\nu + \sum\nolimits_{j\geq 1} F^\mu_j (F^\nu_j)^*\big) v^\nu \Big\rangle \ \leq 0,$$

for all $v = (v^\mu)_{\mu\geq 0} \in \mathfrak{h}\otimes(\mathbb{C}\oplus\mathsf{k}) = \bigoplus_{\mu\geq 0}\mathfrak{h}$ ([Fag], [MoP]), cf. the equalities (1.3) for the unitary case. However in this case the convergence of the Trotter products is only assured in the weak operator topology (or rather in the hybrid norm $\mathcal{F}_{\mathsf{k}}$-weak operator topology, see [LW$_2$]).

Using an extension of the standard Trotter product formula to products of several semigroups, our QS product formula extends to cover a finite number of QS unitary (or contraction) cocycles $(U^1_t)_{t\geq 0}, \dots$, $(U^p_t)_{t\geq 0}$. The coefficient matrix for the QS differential equation of the resulting QS cocycle will then have the block matrix form:

$$\begin{bmatrix} {}^{(1)}K + \cdots + {}^{(p)}K & {}^{(1)}M & {}^{(2)}M & \cdots & {}^{(p)}M \\ {}^{(1)}L & {}^{(1)}W - {}^{(1)}I & 0 & \cdots & 0 \\ {}^{(2)}L & 0 & {}^{(2)}W - {}^{(2)}I & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ {}^{(p)}L & 0 & \cdots & 0 & {}^{(p)}W - {}^{(p)}I \end{bmatrix}.$$

Example. The cocycles considered in [PS$_1$] are the random unitaries defined by

$$U^l(s,t,\omega^l) = e^{i(\omega^l(t)-\omega^l(s))H_l}, \quad 0\leq s\leq t,$$

for $l = 1,2$, where ω^1 and ω^2 are paths of two independent classical Brownian motions $(B^1_t)_{t\geq 0}$ and $(B^2_t)_{t\geq 0}$, and H_1 and H_2 are selfadjoint operators on a Hilbert space $\mathfrak{h}$. Recall the notation (2.3). By viewing $\omega := (\omega^1,\omega^2)$ as a path of the two-dimensional Brownian motion $((B^1_t, B^2_t))_{t\geq 0}$ with probability space Ω, and

$$U^{(1,2)}(s,t,\omega) := e^{i(\omega^1(t)-\omega^1(s))H_1}\, e^{i(\omega^2(t)-\omega^2(s))H_2}, \quad 0\leq s\leq t,$$

as multiplication operators on $L^2(\Omega;\mathfrak{h})$, it is shown—under the assumption that the nonnegative symmetric operator $(H_1)^2 + (H_2)^2$ is selfadjoint—that the sequence $(U^{(1,2)}_n(s,t,\omega))_{n\geq 1}$ of unitary operators:

$$U^{(1,2)}(s,s^n_1,\omega)\big(U^{(1,2)}(s^n_1,s^n_2,\omega)\cdots U^{(1,2)}(t^n_{-1},t^n_0,\omega)\big)U^{(1,2)}(t^n_0,t,\omega)$$

weak-operator converges to the unique contraction-operator valued process satisfying the classical stochastic differential equations

$$\begin{aligned} d_t\, U(s,t,\omega)v = i\, U(s,t,\omega)H_1 v\, dB^1_t(\omega) + i\, U(s,t,\omega)H_2 v\, dB^2_t(\omega) \\ - \tfrac{1}{2}\, U(s,t,\omega)\big((H_1)^2 + (H_2)^2\big) v\, dt \end{aligned}$$

($v \in \mathrm{Dom}((H_1)^2 + (H_2)^2)$), and that if the process $(U(s,t,\omega))_{0\leq s\leq t}$ is unitary-valued then the convergence is strong.

Remark. Under the assumption of selfadjointness of $\sum_{l=1}^d (H_l)^2$, the corresponding result is shown to hold for any finite number of such unitary cocycles $(U^l(s,t,\omega))_{t\geq 0}$, $l = 1,\dots,d$.

This may be recast in our quantum stochastic setting by identifying the Brownian motion $(B^l_t)_{t\geq 0}$ with the quantum stochastic process $(Q^l_t := (A^{l*}_t + A^l_t)^-)_{t\geq 0}$ on $\mathcal{F}^{(l)}$ (where the bar denotes operator closure), and setting

$$U^l_t = e^{iH_l(t)}, \quad t \geq 0,$$

where $H_l(t)$ is the selfadjoint operator $H_l \otimes Q^{(l)}_t$ on $\mathfrak{h} \otimes \mathcal{F}^{(l)}$, for $l = 1, \cdots, d$. Here however the coefficients of the corresponding differential equation are unbounded, with coefficients having block matrix form

$$^{(l)}F = \begin{bmatrix} -\frac{1}{2}(H_l)^2 & iH_l \\ iH_l & 0 \end{bmatrix}, \quad l = 1, \cdots, d,$$

and

$$F = \begin{bmatrix} -\frac{1}{2}K & iH_1 & \cdots & iH_d \\ iH_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ iH_d & 0 & \cdots & 0 \end{bmatrix} \text{ where } K = (H_1)^2 + \cdots (H_d)^2.$$

This class of example is discussed in more detail in [LS$_2$].

4. Concluding remarks

The methods of this paper extend to more general QS cocycles. Firstly, quantum stochastic Trotter product formulae may be obtained for completely contractive QS cocycles on operators spaces and completely positive QS cocycles on C^*-algebras. Secondly, *strongly continuous* (as opposed to Markov-regular) QS cocycles may be shown to satisfy the QS Trotter product formula developed here. Conversely, the formula may be used to construct QS cocycles from simpler cocycles with lower dimensional noises. This yields potential applications to multidimensional diffusions. The basic conditions under which Trotter products converge is that the sum of sufficiently many pairs of associated semigroup generators are pregenerators of contraction semigroups. Here the assumption of analyticity of the expectation semigroups of the constituent cocycles helps ([LS$_1$]). As in the Markov-regular case, strong (as opposed to weak) operator convergence holds for Trotter products of isometric QS cocycles if and only if the limiting cocycle is isometric. Coisometry, on the other hand, is equivalent to isometry of the *dual* cocycle (see [L]). Unitarity for strongly continuous contraction cocycles is assured when the cocycle satisfies a QS differential differential whose coefficients satisfy *Feller conditions* (see [SiG]).

All these extensions are treated in [LS$_2$]. They are facilitated by characterisations of QS cocycles in terms of (a small number of) their associated semigroups ([AcK], [LW$_3$]). Here Skeide's multidimensional generalisation ([Ske]) of a theorem of Parthasarathy and Sunder ([PSu]) plays a key role. The homomorphic property of Trotter product limits of Evans-Hudson type cocycles on operator algebras is tackled in [DGS].

Acknowledgements

Both authors acknowledge support from the UK-India Education and Research Initiative (UKIERI). KBS is grateful for the support of a Bhatnagar Fellowship from CSIR, India, and JML is grateful for the hospitality of the Delhi Centre of the Indian Statistical Institute, where this collaboration began, and the J.N. Centre for Advanced Research, Bangalore.

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